Density of rational points on rational elliptic surfaces

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(Rational) elliptic surfaces and where to find them:

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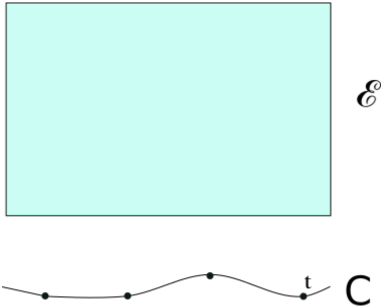

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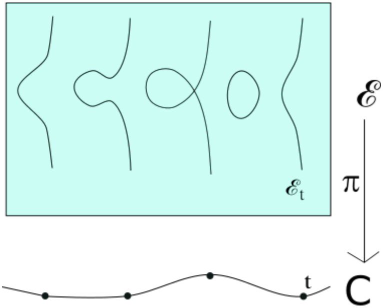

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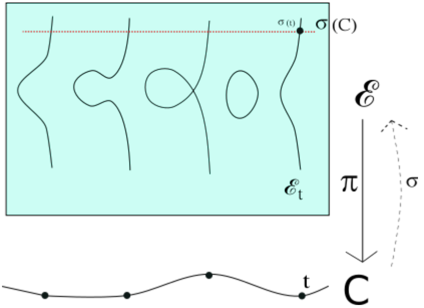

(Rational) elliptic surfaces and where to find them:

An elliptic surface with base $\mathbb{P}^1$, is

- ⋆ $\mathscr{E}$ a smooth projective surface,
- ⋆ $\pi : \mathscr{E} \to \mathbb{P}^1$, such that $\forall t \in \mathbb{P}^1$ the *fiber* $\mathscr{E}_t = \pi^{-1}(t)$ is a smooth genus 1 curve, except for finitely many t,
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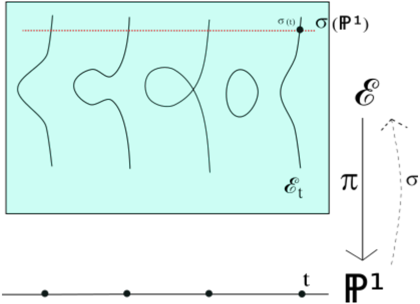

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- $\mathscr{E}$ admits a Weierstrass equation

$$y^2 = x^3 + A(T)x + B(T), \text{ where } A, B \in \mathbb{Z}[T].$$

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- If $\deg A \leq 4$, $\deg B \leq 6$, and $\Delta(T) \notin \mathbb{Q}$ then $\mathscr{E}$ is *rational* (birational to $\mathbb{P}^2$).

Rational elliptic surfaces and where to find them:

Theorem (Iskovskih '79)

Let $\mathscr{E} \to \mathbb{P}^1$ be a rational elliptic surface. Its minimal model $X/\mathbb{Q}$ is:

- *either a conic bundle of degree ≥ 1,*
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The density is shown if $X(\mathbb{Q})$ is non-empty and if X is

- ⋆ a conic bundle of degree ≥ 1 (Kollár - Mella)
- ⋆ a DP of degree ≥ 3 (Segre - Manin)
- ⋆ a DP of degree 2 with a rational point outside of the exceptional curves and of a certain quartic (Salgado, Testa, Várilly-Alvarado).
- ⋆ For DP1, partial results (Ulas, Salgado - van Luijk, Várilly-Alvarado).

Zariski-density and how to prove it

Known fact

Let K be a number field.

$$\#\{t \in \mathbb{P}^1 \mid \mathrm{rk}(\mathscr{E}_t) \neq 0\} = \infty \Leftrightarrow \mathscr{E}(K) \text{ is dense}$$

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Two approaches:

- geometric (computation of the rank, section of infinite order)
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Define two sets $W_{\pm}(\mathscr{E}) = \{t \in \mathbb{P}^1 \mid W(\mathscr{E}_t) = \pm 1\}$.

If the parity conjecture holds on the fibers of $\mathscr{E}$, we have

$$\# W_{-}(\mathscr{E}) = \infty \Rightarrow \mathscr{E}(\mathbb{Q}) \text{ is dense}$$

A conjecture and why to believe it

Conjecture [D.]

Let $\mathscr{E} \to \mathbb{P}^1$ be an elliptic surface over $\mathbb{Q}$. Then $\mathscr{E}(\mathbb{Q})$ is Zariski-dense unless there is an elliptic curve E_0 such that $\mathscr{E} \simeq E_0 \times \mathbb{P}^1$.

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- For non-isotrivial elliptic surface: believed.
- Isotrivial elliptic surface: new!

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Identify finite places v of $\mathbb{Q}(T)$ with corresponding monic irreducible $P_v \in \mathbb{Z}[X]$.

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Let $\mathscr{E} \to \mathbb{P}^1$ be a non-isotrivial elliptic surface over $\mathbb{Q}$. Assume

- ⋆ **Chowla's conjecture** for $M_{\mathscr{E}} = \prod_{v \text{ mult}} P_v$ and
- ⋆ **Squarefree conjecture** for all $P_{v'}$ of bad reduction.

Then

$$\# W_\pm = \{t \in \mathbb{P}^1_{\mathbb{Q}} \mid W(\mathscr{E}) = \pm 1\} = \infty.$$

Moreover, can avoid assuming **squarefree conjecture** if P_v of additive potentially good reduction satisfy a technical hypothesis.

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- Uses new sieve: combining Chowla's and squarefree conjectures

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- For isotrivial $\mathscr{E}$, it happens that the root number $t \mapsto W(\mathscr{E}_t)$ is constant. [Cassels-Schinzel '82]: On

$$\mathscr{E} : y^2 = x^3 - 7^2(t^4+1)^2 x,$$

one has $W(\mathscr{E}_t) = -1$ for every $t \in \mathbb{Q}$.

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Theorem (D.)

Let $\mathscr{E} \to \mathbb{P}^1$ be a rational elliptic surface. Then $\mathscr{E}(\mathbb{Q})$ is dense in the following cases.

- ⋆ $\mathscr{E}$ isotrivial and $j(T) \neq 0$
- ⋆ $\mathscr{E}$ admits a place of type II^*, III^*, IV^* or I_m^* $(m \geq 1)$.

Sketch of proof

If $j \neq 0, 1728$, then X the minimal model of $\mathscr{E}$ is a conic bundle:
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if $j = 1728$, find an "almost section" $Q \in \mathscr{E}_t(\mathbb{Q})$ (only for $t \in \mathbb{Q}$ satisfying a certain property) which is

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  In this case, it is still possible to prove the density using the minimal model:
  - X is never a Del Pezzo of degree 1
  - if X is a DP≥ 3 or a conic bundle $\Longrightarrow$ $\mathscr{E}(\mathbb{Q})$ dense,
  - if X is a DP2, then using the "almost section" Q one can show that there exists a point outside of the exceptional curves and the quartic $\Longrightarrow$ $\mathscr{E}(\mathbb{Q})$ dense.

Missing cases and what is known about them

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- Salgado & Van Luijk: When the minimal model is a DP1 and there is a linear polynomial of type I_1 (among other results)
- Bettin, David & Delaunay: Restrict the degree of coefficients and study the generic rank with Nagao's formula. ($\implies$ project at ICTP!)

Thank you for your attention!